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Meixner and Pollaczek Spaces of Entire Functions

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We study examples of Hilbert spaces whose elements are entire functions and which have these properties:

(H1) Whenever $F(z)$ is in the space and has a nonreal zero w, the function $F(z)(z-\bar{w})/(z-w)$ belongs to the space and has the same norm as $F(z)$.

(H2) For each nonreal number w, the linear functional defined on the space by $F(z) \to F(w)$ is continuous.

(H3) The function $F^*(z) = \bar{F}(\bar{z})$ belongs to the space whenever $F(z)$ belongs to the space and it always has the same norm as $F(z)$.

The theory of such spaces is related to the theory of entire functions $E(z)$ which satisfy the inequality

$$| E(x - iy) | < | E(x + iy) |$$

for $y > 0$. If $E(z)$ is such a function, we write $E(z) = A(z) - iB(z)$ where $A(z)$ and $B(z)$ are entire functions which are real for real z and

$$K(w, z) = \frac{B(z)\bar{A}(w) - A(z)\bar{B}(w)}{\pi(z - \bar{w})}.$$

Let $\mathscr{H}(E)$ be the set of entire functions $F(z)$ such that

$$\| F \|^2 = \int_{-\infty}^{+\infty} \left| \frac{F(t)}{E(t)} \right|^2 dt < \infty$$

and such that

$$| F(z) |^2 \leqslant \| F \|^2 K(z, z)$$

for all complex z. Then $\mathscr{H}(E)$ is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). For each complex number w, $K(w, z)$ belongs to $\mathscr{H}(E)$ as a function of z, and the identity

$$F(w) = \langle F(t), K(w, t) \rangle$$

holds for every element $F(z)$ of $\mathscr{H}(E)$. A Hilbert space, whose elements are entire functions, which satisfies the axioms (H1), (H2), and (H3), and which contains a nonzero element, is equal isometrically to a space $\mathscr{H}(E)$.

Many examples of such spaces can be constructed by the use of hypergeometric functions. The spaces now studied are finite dimensional spaces related to Pollaczek's orthogonal polynomials. They are characterized by an identity involving two parameters, h and ω. It implies a recurrence relation for the defining functions $A(z)$ and $B(z)$ of the space.

THEOREM 1. *Let $\mathscr{H}(E)$ be a given space, and let h and ω be given numbers, $h > 0$, $\omega \neq \bar{\omega}$, and $|\omega| = 1$. Assume that the functions*

$$(h - iz)\,[F(z+i) - F(z)] \qquad \text{and} \qquad (h + iz)\,[F(z-i) - F(z)]$$

belong to the space whenever $F(z)$ belongs to the space and that the identity

$$\langle \bar{\omega}(h - it)F(t+i), G(t)\rangle + \langle \omega(h+it)F(t), G(t+i)\rangle = 0$$

holds for every element $G(z)$ of the space when $F(z)$ belongs to the domain of multiplication by z in the space. Then there exist real numbers u_+, v_+, u_-, v_- such that the functions

$$S_+(z) = A(z)\,u_+ + B(z)\,v_+ \qquad \text{and} \qquad S_-(z) = A(z)\,u_- + B(z)\,v_-$$

are linearly independent and satisfy the recurrence relations

$$\begin{aligned}
\bar{\omega}(h - iz)\,S_+(z+i) + i(\omega + \bar{\omega})\,zS_+(z) - \omega(h+iz)\,S_+(z-i) \\
= \lambda_+(\bar{\omega} - \omega)\,S_+(z), \\
\bar{\omega}(h - iz)\,S_-(z+i) + i(\omega + \bar{\omega})\,zS_-(z) - \omega(h+iz)\,S_-(z-i) \\
= \lambda_-(\bar{\omega} - \omega)\,S_-(z), \\
(h - iz)\,S_+(z+i) + 2izS_+(z) - (h+iz)\,S_+(z-i) \\
= (\lambda_+ - h)\,(\bar{\omega} - \omega)\,S_-(z), \\
\bar{\omega}^2(h - iz)\,S_-(z+i) + 2izS_-(z) - \omega^2(h+iz)\,S_-(z-i) \\
= (\lambda_- + h)\,(\bar{\omega} - \omega)\,S_+(z)
\end{aligned}$$

for some real numbers λ_+ and λ_- such that $\lambda_+ = 1 + \lambda_-$.

Using this information we can construct all the spaces which satisfy the hypotheses of Theorem 1.

THEOREM 2. *Let h and ω be given numbers, $h > 0$, $\omega \neq \bar{\omega}$, and $|\omega| = 1$. Then the polynomials $\Phi_n(z)$ defined by*

$$\Phi_n(z) = \bar{\omega}^n F(-n, h + iz; 2h; 1 - \omega^2)$$

are real for real z and satisfy the identities

$$\begin{aligned}
&\bar{\omega}(h - iz)\,\Phi_n(z + i) + i(\omega + \bar{\omega})\,z\Phi_n(z) - \omega(h + iz)\,\Phi_n(z - i)\\
&\qquad = (h + n)\,(\bar{\omega} - \omega)\,\Phi_n(z),\\
&(h - iz)\,\Phi_n(z + i) + 2iz\Phi_n(z) - (h + iz)\,\Phi_n(z - i)\\
&\qquad = n(\bar{\omega} - \omega)\,\Phi_{n-1}(z),\\
&\bar{\omega}^2(h - iz)\,\Phi_n(z + i) + 2iz\Phi_n(z) - \omega^2(h + iz)\,\Phi_n(z - i)\\
&\qquad = (2h + n)\,(\bar{\omega} - \omega)\,\Phi_{n+1}(z),\\
&iz(\omega - \bar{\omega})\,\Phi_n(z) = n\Phi_{n-1}(z) - (h + n)\,(\omega + \bar{\omega})\,\Phi_n(z) + (2h + n)\,\Phi_{n+1}(z).
\end{aligned}$$

There exist spaces $\mathscr{H}(E_n)$, $n = 1, 2, 3, \ldots$, satisfying the hypotheses of Theorem 1, such that $\mathscr{H}(E_n)$ is contained isometrically in $\mathscr{H}(E_{n+1})$ for every n, such that $\Phi_0(z)$ spans $\mathscr{H}(E_1)$, and such that $\Phi_n(z)$ spans the orthogonal complement of $\mathscr{H}(E_n)$ in $\mathscr{H}(E_{n+1})$ for every $n > 0$. The spaces can be chosen so that

$$\| \Phi_n(t) \|^2 = \frac{\Gamma(1 + n)\,\Gamma(2h)}{\Gamma(2h + n)}$$

for every n. If $-\omega^2 = \exp(2i\theta)$, $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, then

$$\pi 2^{1-2h}\Gamma(2h)\sec^{2h}(\theta)\,\| F(t) \|^2 = \int_{-\infty}^{+\infty} | F(t)\, e^{-\theta t}\Gamma(h - it) |^2\, dt$$

for every polynomial $F(z)$.

THEOREM 3. *If $\mathscr{H}(E)$ is a space which satisfies the hypotheses of Theorem 1, then there exists an entire function $S(z)$ which is real for real z, has no zeros, and is periodic of period i, and there exists an index r in Theorem 2 such that the transformation $F(z) \to S(z)F(z)$ is an isometry of $\mathscr{H}(E_r)$ onto $\mathscr{H}(E)$.*

The spaces of Theorem 1 are related to a space of square summable power series. Let a, b, and c be numbers such that the coefficients of the hypergeometric series

$$F(a, b; c; z) = 1 + \frac{ab}{1!c}\,z + \frac{a(a + 1)\,b(b + 1)}{2!c(c + 1)}\,z^2 + \cdots$$

are all positive. By $\mathscr{C}(a, b; c; z)$ we mean the Hilbert space of power series $f(z) = \sum a_n z^n$ with complex coefficients such that

$$\|f(z)\|^2 = | a_0 |^2 + \frac{1!c}{ab}\,| a_1 |^2 + \frac{2!c(c + 1)}{a(a + 1)\,b(b + 1)}\,| a_2 |^2 + \cdots < \infty.$$

The series which belong to $\mathscr{C}(a, b; c; z)$ converge and represent analytic functions in the unit disk. The series $F(a, b; c; \bar{w}z)$ belongs to the space when $|w| < 1$, and the identity

$$f(w) = \langle f(z), F(a, b; c; \bar{w}z) \rangle$$

holds for every element $f(z)$ of the space.

THEOREM 4. *In Theorem 2 if $f(z) = \sum a_n z^n$ is a polynomial of degree less than r, then its eigentransform $F(z) = \sum a_n \Phi_n(z)$ belongs to $\mathscr{H}(E_r)$ and*

$$\int_{-\infty}^{+\infty} \left| \frac{F(t)}{E_r(t)} \right|^2 dt = \| f(z) \|^2$$

where the norm of $f(z)$ is taken in $\mathscr{C}(2h, 1; 1; z)$. Every element of $\mathscr{H}(E_r)$ is of this form. The identity

$$\Gamma(h - iz)\, \Gamma(h + iz) F(z)$$

$$= 2^{1-2h} \Gamma(2h) \int_{-1}^{1} (1 - t)^{h+iz-1} (1 + t)^{h-iz-1} f(\tfrac{1}{2}\, \omega + \tfrac{1}{2}\, \bar{\omega} - \tfrac{1}{2}\, t\omega + \tfrac{1}{2}\, t\bar{\omega})\, dt$$

holds for $-h < y < h$ whenever $f(z)$ is a polynomial and $F(z)$ is its eigentransform. Let $f(z)$ and $g(z)$ be polynomials and let $F(z)$ and $G(z)$ be their eigentransforms. The condition

$$(\bar{\omega} - \omega)\, G(z) = \bar{\omega}(h - iz) F(z + i) + i(\omega + \bar{\omega})\, zF(z) - \omega(h + iz) F(z - i)$$

is necessary and sufficient that

$$g(z) = hf(z) + zf'(z).$$

The condition

$$(\bar{\omega} - \omega)\, G(z) = (h - iz) F(z + i) + 2izF(z) - (h + iz) F(z - i)$$

is necessary and sufficient that

$$g(z) = f'(z).$$

The condition

$$(\bar{\omega} - \omega)\, G(z) = \bar{\omega}^2(h - iz) F(z + i) + 2izF(z) - \omega^2(h + iz) F(z - i)$$

is necessary and sufficient that

$$g(z) = 2hzf(z) + z^2 f'(z).$$

The condition

$$G(z) = i(\omega - \bar{\omega})\, zF(z)$$

is necessary and sufficient that

$$g(z) = h(2z - \omega - \bar{\omega}) f(z) + (z^2 - \omega z - \bar{\omega} z + 1) f'(z).$$

A similar family of spaces is associated with Meixner's orthogonal polynomials.

THEOREM 5. *Let $\mathscr{H}(E)$ be a given space, and let h and ω be given positive numbers, $\omega \neq \omega^{-1}$. Assume that the functions $(h+z)[F(z+1)-F(z)]$ and $(h-z)[F(z-1)-F(z)]$ belong to the space whenever $F(z)$ belongs to the space and that the identity*

$$\langle \omega^{-1}(h-t)F(t-1), G(t)\rangle + \langle \omega(h+t)F(t), G(t+1)\rangle = 0$$

holds for every element $G(z)$ of the space when $F(z)$ belongs to the domain of multiplication by z in the space. Then there exist real numbers u_+, v_+, u_-, v_- such that the functions

$$S_+(z) = A(z)\,u_+ + B(z)\,v_+ \qquad \text{and} \qquad S_-(z) = A(z)\,u_- + B(z)\,v_-$$

are linearly independent and satisfy the recurrence relations

$$\omega(h+z)\,S_+(z+1) - (\omega+\omega^{-1})\,zS_+(z) - \omega^{-1}(h-z)\,S_+(z-1) = \lambda_+(\omega-\omega^{-1})\,S_+(z),$$

$$\omega(h+z)\,S_-(z+1) - (\omega+\omega^{-1})\,zS_-(z) - \omega^{-1}(h-z)\,S_-(z-1) = \lambda_-(\omega-\omega^{-1})\,S_-(z),$$

$$(h+z)\,S_+(z+1) - 2zS_+(z) - (h-z)\,S_+(z-1) = (\lambda_+ - h)(\omega-\omega^{-1})\,S_-(z),$$

$$\omega^2(h+z)\,S_-(z+1) - 2zS_-(z) - \omega^{-2}(h-z)\,S_-(z-1) = (\lambda_- + h)(\omega-\omega^{-1})\,S_+(z)$$

for some real numbers λ_+ and λ_- such that $\lambda_+ = 1 + \lambda_-$.

We can now construct all the spaces which satisfy the hypotheses of Theorem 5.

THEOREM 6. *Let h and ω be given positive numbers, $\omega \neq \omega^{-1}$. Then the polynomials $\Phi_n(z)$ defined by*

$$\Phi_n(z) = \omega^{-n}F(-n, h+z; 2h; 1-\omega^2)$$

are real for real z and satisfy the identities

$$\omega(h+z)\,\Phi_n(z+1) - (\omega+\omega^{-1})\,z\Phi_n(z) - \omega^{-1}(h-z)\,\Phi_n(z-1) = (h+n)(\omega-\omega^{-1})\,\Phi_n(z),$$

$$(h+z)\,\Phi_n(z+1) - 2z\Phi_n(z) - (h-z)\,\Phi_n(z-1) = n(\omega-\omega^{-1})\,\Phi_{n-1}(z),$$

$$\omega^2(h+z)\,\Phi_n(z+1) - 2z\Phi_n(z) - \omega^{-2}(h-z)\,\Phi_n(z-1)$$
$$= (2h+n)\,(\omega - \omega^{-1})\,\Phi_{n+1}(z),$$

$$z(\omega - \omega^{-1})\,\Phi_n(z) = n\Phi_{n-1}(z) - (h+n)\,(\omega+\omega^{-1})\,\Phi_n(z) + (2h+n)\,\Phi_{n+1}(z).$$

There exist spaces $\mathscr{H}(E_n)$, $n = 1, 2, 3, \ldots$, satisfying the hypotheses of Theorem 5, such that $\mathscr{H}(E_n)$ is contained isometrically in $\mathscr{H}(E_{n+1})$ for every n, such that $\Phi_0(z)$ spans $\mathscr{H}(E_1)$, and such that $\Phi_n(z)$ spans the orthogonal complement of $\mathscr{H}(E_n)$ in $\mathscr{H}(E_{n+1})$ for $n > 0$. The spaces can be chosen so that

$$\| \Phi_n(t) \|^2 = \frac{\Gamma(1+n)\,\Gamma(2h)}{\Gamma(2h+n)}$$

for every n. The identity

$$\Gamma(2h)\,(1-\omega^2)^{-2h}\,\| F(t) \|^2 = \sum_{n=0}^{\infty} \frac{| F(h+n) |^2\,\omega^{2n}\Gamma(2h+n)}{\Gamma(1+n)}$$

holds for every polynomial $F(z)$ when $\omega < 1$. The corresponding identity for $\omega > 1$ is

$$\Gamma(2h)\,(1-\omega^{-2})^{-2h}\,\| F(t) \|^2 = \sum_{n=0}^{\infty} \frac{| F(-h-n) |^2\,\omega^{-2n}\Gamma(2h+n)}{\Gamma(1+n)}.$$

THEOREM 7. *If $\mathscr{H}(E)$ is a space which satisfies the hypotheses of Theorem 5, then there exists an entire function $S(z)$ which is real for real z, has only real zeros, and is periodic of period 1, and there is an index r in Theorem 6 such that the transformation $F(z) \to S(z)F(z)$ is an isometry of $\mathscr{H}(E_r)$ onto $\mathscr{H}(E)$.*

These spaces are also related to generalized spaces of square summable power series.

THEOREM 8. *In Theorem 6 if $f(z) = \sum a_n z^n$ is a polynomial of degree less than r, then its eigentransform $F(z) = \sum a_n \Phi_n(z)$ belongs to $\mathscr{H}(E_r)$ and*

$$\int_{-\infty}^{+\infty} \left| \frac{F(t)}{E_r(t)} \right|^2 dt = \| f(z) \|^2$$

where the norm of $f(z)$ is taken in $\mathscr{C}(2h, 1; 1; z)$. Every element of $\mathscr{H}(E_r)$ is of this form. The identity

$$\Gamma(h-z)\,\Gamma(h+z)\,F(z)$$
$$= 2^{1-2h}\Gamma(2h) \int_{-1}^{1} (1-t)^{h+z-1}\,(1+t)^{h-z-1} f(\tfrac{1}{2}\,\omega + \tfrac{1}{2}\,\omega^{-1} - \tfrac{1}{2}\,t\omega + \tfrac{1}{2}\,t\omega^{-1})\,dt$$

holds for $-h < x < h$ *whenever* $f(z)$ *is a polynomial and* $F(z)$ *is its eigentransform. Let* $f(z)$ *and* $g(z)$ *be polynomials and let* $F(z)$ *and* $G(z)$ *be their eigentransforms. The condition*

$$(\omega - \omega^{-1})\, G(z) = \omega(h + z) F(z + 1) - (\omega + \omega^{-1})\, zF(z) - \omega^{-1}(h - z) F(z - 1)$$

is necessary and sufficient that

$$g(z) = hf(z) + zf'(z).$$

The condition

$$(\omega - \omega^{-1})\, G(z) = (h + z) F(z + 1) - 2zF(z) - (h - z) F(z - 1)$$

is necessary and sufficient that

$$g(z) = f'(z).$$

The condition

$$(\omega - \omega^{-1})\, G(z) = \omega^2(h + z) F(z + 1) - 2zF(z) - \omega^{-2}(h - z) F(z - 1)$$

is necessary and sufficient that

$$g(z) = 2hzf(z) + z^2 f'(z).$$

The condition

$$G(z) = (\omega - \omega^{-1})\, zF(z)$$

is necessary and sufficient that

$$g(z) = h(2z - \omega - \omega^{-1}) f(z) + (z^2 - \omega z - \omega^{-1} z + 1) f'(z).$$

Information about Meixner's polynomials can be obtained from Meixner [1] and the Bateman manuscript project [2]. Pollaczek's polynomials are treated by Pollaczek [3], Szegö [4], and the Bateman manuscript project [2].

PROOF OF THEOREM 1. Since the transformations

$$F(z) \to (h - iz)\,[F(z + i) - F(z)] \quad \text{and} \quad F(z) \to (h + iz)\,[F(z - i) - F(z)]$$

are everywhere defined in the space and have a closed graph, they are bounded. Let L_+, L_-, and D be the transformations on entire functions defined by $D : F(z) \to G(z)$ if

$$(\bar{\omega} - \omega)\, G(z) = \bar{\omega}(h - iz) F(z + i) + i(\omega + \bar{\omega})\, zF(z) - \omega(h + iz) F(z - i),$$

$L_- : F(z) \to G(z)$ if

$$(\bar{\omega} - \omega)\, G(z) = (h - iz) F(z + i) + 2izF(z) - (h + iz) F(z - i),$$

and $L_+ : F(z) \to G(z)$ if

$$(\bar{\omega} - \omega)\, G(z) = \bar{\omega}^2(h - iz) F(z + i) + 2izF(z) - \omega^2(h + iz) F(z - i).$$

A straightforward calculation will show that they satisfy the commutator identities

$$L_-L_+ - L_+L_- = 2D,$$
$$DL_- - L_-D = -L_-,$$
$$DL_+ - L_+D = L_+.$$

By hypothesis, D is an everywhere defined transformation in the space. The hypotheses imply that L_- takes the space into itself and is bounded in the space. On the other hand, L_+ takes $F(z)$ into the space whenever $F(z)$ belongs to the domain of multiplication by z in the space. Note that D takes the domain of multiplication by z into itself. For if $F(z)$ belongs to the domain of multiplication by z and if $D : F(z) \to G(z)$, then

$$D : zF(z) \to zG(z) + i\bar{\omega}(h - iz)F(z + i) + i\omega(h + iz)F(z - i).$$

By $(H3)$ the hypotheses imply that the identity

$$\langle \omega(h + it)F(t - i), G(t)\rangle + \langle \bar{\omega}(h - it)F(t), G(t - i)\rangle = 0$$

holds for every element $G(z)$ of the space when $F(z)$ belongs to the domain of multiplication by z in the space. The hypotheses also imply that the domain of multiplication by z in the space contains $F(z + i) - F(z)$ and $F(z - i) - F(z)$ whenever $F(z)$ belongs to the space. Apply the last identity with $F(z)$ replaced by $F(z + i) - F(z)$, and the identity of the theorem with $F(z)$ replaced by $F(z - i) - F(z)$. A straightforward calculation will show that D is self-adjoint. A similar argument shows that the identity

$$\langle L_-F, G\rangle = \langle F, L_+G\rangle$$

holds for every $F(z)$ when $G(z)$ is in the domain of multiplication by z in the space. By Theorem 29 of [5], the orthogonal complement of the domain of multiplication by z has dimension zero or one. Since L_- is a bounded transformation in the space, we can conclude that L_+ is bounded on the domain of multiplication by z in the space. It follows that multiplication by z is a bounded transformation in the space. Multiplication by z is not an everywhere defined transformation in the space since this would contradict (H2) when $|w|$ is greater than the bound of multiplication by z. Since multiplication by z is bounded and has a closed graph, its domain is closed. Since the domain is not the full space, there exists a nonzero element $S(z)$ of the space which is orthogonal to the domain of multiplication by z. By Theorem 22 of [5], the space has an orthogonal basis consisting of the function $S(z)$ and the functions of the form $S(z)/(z - t_n)$ which belong to the space. Since

$$\frac{zS(z)}{z - t_n} = S(z) + \frac{t_nS(z)}{z - t_n}$$

and since multiplication by z is bounded, the numbers (t_n) are bounded. Since they are the zeros of a nonzero entire function, they are a finite set. It follows that the space is finite dimensional.

Assume that the space $\mathscr{H}(E) = \mathscr{H}(E_r)$ has dimension r. Since there exists an element of the space which has a nonzero value at any given nonreal point, the transformation $F(z) \to (h - iz)[F(z+i) - F(z)]$ does not take the space onto itself. Since the space is finite dimensional, the transformation has a nonzero kernel. It follows that there exists a nonzero element of the space which is periodic of period i. Since the space is finite dimensional, any nonzero element of the space has only a finite number of nonreal zeros. Since $S(z)$ is periodic of period i, it has no zeros. By Problem 88 of [5], $S(z)$ and $S^*(z)$ are linearly dependent and the elements of the space are the functions of the form $P(z)\, S(z)$ where $P(z)$ is a polynomial of degree less than r. In what follows we assume that $S(z)$ is chosen of norm one and real for real z.

Let $S_0(z)$, $S_1(z)$, $S_2(z)$,..., be the entire functions defined inductively by

$$S_0(z) = S(z) \qquad \text{and} \qquad L_+ : S_n(z) \to (2h + n)\, S_{n+1}(z).$$

The commutator identities for L_+, L_-, and D imply that

$$D : S_n(z) \to (h + n)\, S_n(z)$$

for every n and that

$$L_- : S_n(z) \to nS_{n-1}(z)$$

when $n > 0$. It follows that the identity

$$iz(\omega - \bar{\omega})\, S_n(z) = nS_{n-1}(z) - (h + n)(\omega + \bar{\omega})\, S_n(z) + (2h + n)\, S_{n+1}(z)$$

is valid when $n > 0$. The same identity holds when $n = 0$ if the term $S_{n-1}(z)$ is omitted. From this we see that $S_n(z)/S(z)$ is a polynomial of degree n for every n. So $S_n(z)$ belongs to $\mathscr{H}(E_r)$ when $n < r$. Since the functions $S_0(z)$,..., $S_{r-1}(z)$ are eigenfunctions of a self-adjoint transformation for distinct eigenvalues, they are orthogonal in $\mathscr{H}(E_r)$. Since L_- and L_+ are adjoint transformations in $\mathscr{H}(E_r)$; the identity

$$\langle L_+ S_{n-1}, S_n \rangle = \langle S_{n-1}, L_- S_n \rangle$$

holds when $0 < n < r$. It follows that

$$(2h + n - 1)\, \| S_n(t) \|^2 = n \, \| S_{n-1}(t) \|^2.$$

Since we assume that $\| S_0(t) \| = 1$, we can conclude that

$$\| S_n(t) \|^2 = \frac{\Gamma(1 + n)\, \Gamma(2h)}{\Gamma(2h + n)}.$$

Since $S_0(z),\ldots, S_{r-2}(z)$ belong to the domain of multiplication by z in the space, $S_{r-1}(z)$ spans the orthogonal complement of the domain of multiplication by z in the space. By Theorem 29 of [5], $S_{r-1}(z)$ is of the form $A_r(z)\, u_- + B_r(z)\, v_-$ for some real numbers u_- and v_-.

To obtain the desired form of $S_r(z)$, we construct an $(r+1)$-dimensional space which contains $\mathscr{H}(E_r)$ isometrically, such that $S_r(z)$ belongs to the space and spans the orthogonal complement of $\mathscr{H}(E_r)$. The norm of $S_r(z)$ is defined by the last formula with $n = r$. It is easily verified that multiplication by z is a symmetric transformation in the extended space and that its domain contains $\mathscr{H}(E_r)$. It follows that the extended space satisfies (H1). The axiom (H2) is satisfied because the space is finite dimensional. The axiom (H3) is satisfied because the functions $S_n(z)$ are real for real z. By Theorem 23 of [5], the extended space is a space $\mathscr{H}(E_{r+1})$. By Problem 87 of [5], we can choose $E_{r+1}(z)$ so that

$$(A_{r+1}(z), B_{r+1}(z)) = (A_r(z), B_r(z)) \begin{pmatrix} 1-\beta z & \alpha z \\ -\gamma z & 1+\beta z \end{pmatrix}$$

for some real numbers α, β, γ such that $\alpha \geqslant 0$, $\gamma \geqslant 0$, and $\alpha\gamma = \beta^2$. Since $S_r(z)$ spans the orthogonal complement of $\mathscr{H}(E_r)$ in $\mathscr{H}(E_{r+1})$, there exist real numbers u_+ and v_+ such that

$$S_r(z) = A_{r+1}(z)\, u_+ + B_{r+1}(z)\, v_+ \,.$$

It follows that

$$S_r(z) = A_r(z)\, u_+ + B_r(z)\, v_+ \,.$$

The theorem follows with $S_+(z) = S_r(z)$ and $S_-(z) = S_{r-1}(z)$.

Proof of Theorem 2. It is immediate from the definition of the hypergeometric series that $F(-n, h+iz; 2h; 1-\omega^2)$ is a polynomial of degree n. The stated identities for $\Phi_n(z)$ follow from Gauss's relations for contiguous hypergeometric functions, Erdélyi [2]. Since $\Phi_0(z) = 1$, it follows inductively from these identities that each function $\Phi_n(z)$ is real for real z. Since each polynomial $\Phi_n(z)$ is of degree n, every polynomial of degree n can be uniquely written as a linear combination of $\Phi_0(z),\ldots, \Phi_n(z)$. It follows that there exists a unique inner product on polynomials with respect to which the functions $\Phi_n(z)$ are an orthogonal set, such that

$$\| \Phi_n(t) \|^2 = \frac{\Gamma(1+n)\,\Gamma(2h)}{\Gamma(2h+n)}$$

for every n. If L_+, L_-, and D are defined as in the proof of Theorem 1, then it is easily verified that the identities

$$\langle DF, G\rangle = \langle F, DG\rangle,$$
$$\langle L_+F, G\rangle = \langle F, L_-G\rangle,$$
$$\langle tF(t), G(t)\rangle = \langle F(t), tG(t)\rangle$$

are valid for all polynomials $F(z)$ and $G(z)$. For every $r = 1, 2, 3,...,$ the polynomials of degree less than r are a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). By Theorem 23 of [5], this space is equal isometrically to a space $\mathscr{H}(E_r)$. The space satisfies the hypotheses of Theorem 1 since the transformations

$$F(z) \to (h - iz)[F(z + i) - F(z)] \quad \text{and} \quad F(z) \to (h + iz)[F(z - i) - F(z)]$$

do not raise the degrees of polynomials and since the last three identities are valid.

To prove the last formula of the theorem, consider the set of all polynomials with the new inner product

$$\langle F(t), G(t)\rangle_1 = \int_{-\infty}^{+\infty} F(t)\, \bar{G}(t)\, e^{-2\theta t}\Gamma(h - it)\, \Gamma(h + it)\, dt.$$

Absolute convergence of the integral follows from the identity

$$\Gamma(\tfrac{1}{2} - iz)\, \Gamma(\tfrac{1}{2} + iz) = \pi \operatorname{sech}(\pi z)$$

when $h = \frac{1}{2}$. When $h - \frac{1}{2}$ is an integer, convergence follows from the recurrence relation $z\Gamma(z) = \Gamma(z + 1)$ for the gamma function. In the general case convergence follows because $|\Gamma(1 - iz)|$ is a nondecreasing function of $y > 0$ for each fixed x, by Problem 243 of [5]. From this we also see that if ϵ is any given positive number, $\epsilon < \frac{1}{2}\pi - |\theta|$, then

$$\lim_{|x|\to\infty} e^{\epsilon|x|}\, |e^{-\theta z}\Gamma(h - iz)| = 0$$

uniformly in the strip $0 \leqslant y \leqslant 1$. The identity

$$\begin{aligned}
&\langle \bar{\omega}(h - it)F(t + i), G(t)\rangle_1 \\
&= \bar{\omega} \int_{-\infty}^{+\infty} F(t + i)\, \bar{G}(t)\, e^{-2\theta t}\Gamma(h - it + 1)\, \Gamma(h + it)\, dt \\
&= -\,\omega \int_{-\infty}^{+\infty} F(t)\, \bar{G}(t + i)\, e^{-2\theta t}\Gamma(h - it)\, \Gamma(h + it + 1)\, dt \\
&= -\,\langle F(t), \bar{\omega}(h - it)\, G(t + i)\rangle_1
\end{aligned}$$

is obtained by applying Cauchy's formula in the rectangle whose vertices are $-n, n, n + i, -n + i$ and letting $n \to \infty$. By conjugating each side of this equation, we obtain the identity

$$\langle \omega(h + it)F(t - i), G(t)\rangle_1 = -\,\langle F(t), \omega(h + it)\, G(t - i)\rangle_1 .$$

It follows that D acts as a self-adjoint transformation on polynomials of degree at most r when they are considered in the new inner product. From the orthogonal sets constructed in the proof of Theorem 1, it follows that there exists a constant κ such that

$$\langle F(t), G(t)\rangle_1 = \kappa \langle F(t), G(t)\rangle$$

for all polynomials $F(z)$ and $G(z)$. On taking $F(z) = G(z) = 1$, we find that

$$\kappa = \int_{-\infty}^{+\infty} e^{-2\theta t} \Gamma(h - it)\, \Gamma(h + it)\, dt.$$

To evaluate the integral we use the identities

$$\lambda^{iz} \Gamma(h - iz) = \lambda^h \int_0^\infty \exp(-\lambda t)\, t^{h-iz-1}\, dt,$$

$$\lambda^{-iz} \Gamma(h - iz) = \lambda^{-h} \int_0^\infty \exp(-\lambda^{-1} t)\, t^{h-iz-1}\, dt,$$

which are valid for all positive values of λ when $y > 0$. By Plancherel's formula for the Mellin transformation,

$$\begin{aligned}\int_{-\infty}^{+\infty} \lambda^{2it} \Gamma(h - it)\, \Gamma(h + it)\, dt &= 2\pi \int_0^\infty \exp(-\lambda t - \lambda^{-1} t)\, t^{2h-1}\, dt \\ &= 2\pi \Gamma(2h)\, (\lambda + \lambda^{-1})^{-2h}.\end{aligned}$$

It follows that

$$\int_{-\infty}^{+\infty} e^{-2\theta t} \Gamma(h - it)\, \Gamma(h + it)\, dt = \pi 2^{1-2h} \Gamma(2h) \sec^{2h}(\theta)$$

for all imaginary values of θ. The formula holds by analytic continuation in the strip $-\frac{1}{2}\pi < \operatorname{Re}\theta < \frac{1}{2}\pi$ if the fractional power is defined so as to be continuous in the strip and positive when θ is real.

PROOF OF THEOREM 3. The theorem follows immediately on comparing the proofs of Theorems 1 and 2.

PROOF OF THEOREM 4. A formula of Euler, Erdélyi [6], states that

$$\Gamma(b)\, \Gamma(c - b) F(a, b; c; z) = \Gamma(c) \int_0^1 t^{b-1} (1 - t)^{c-b-1} (1 - tz)^{-a}\, dt$$

when $\operatorname{Re} c > \operatorname{Re} b > 0$. It follows that

$$\begin{aligned}&\Gamma(h - iz)\, \Gamma(h + iz)\, \Phi_n(z) \\ &= 2^{1-2h} \Gamma(2h) \int_{-1}^1 (1 - t)^{h+iz-1} (1 + t)^{h-iz-1} \left(\tfrac{1}{2}\omega + \tfrac{1}{2}\bar{\omega} - \tfrac{1}{2} t\omega + \tfrac{1}{2} t\bar{\omega}\right)^n dt\end{aligned}$$

for $-h < y < h$. The theorem is now proved by a routine computation using the results of Theorem 2.

PROOF OF THEOREM 5. An explicit proof of the theorem is omitted since it is closely analogous to the proof of Theorem 1. The transformations L_+, L_-, and D are now defined by $D : F(z) \to G(z)$ if

$$(\omega - \omega^{-1})\, G(z) = \omega(h + z) F(z + 1) - (\omega + \omega^{-1})\, zF(z) - \omega^{-1}(h - z) F(z - 1),$$

$L_- : F(z) \to G(z)$ if

$$(\omega - \omega^{-1})\, G(z) = (h + z) F(z + 1) - 2zF(z) - (h - z) F(z - 1),$$

and $L_+ : F(z) \to G(z)$ if

$$(\omega - \omega^{-1})\, G(z) = \omega^2(h + z) F(z + 1) - 2zF(z) - \omega^{-2}(h - z) F(z - 1).$$

PROOF OF THEOREM 6. The proof of the theorem is closely analogous to the proof of Theorem 2 except that it uses the binomial expansion

$$\Gamma(2h)\,(1 - \omega^2)^{-2h} = \sum_{n=0}^{\infty} \frac{\omega^{2n}\Gamma(2h + n)}{\Gamma(1 + n)}$$

when $\omega < 1$.

PROOF OF THEOREM 7. This result follows from the indicated proofs of Theorems 5 and 6.

PROOF OF THEOREM 8. These results follow from Euler's formula as in the proof of Theorem 4.

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